properties of the simple plane electromagnetic wave, you can analyze a surprisingly wide variety of electromagnetic devices, including interferometers, rectangular hollow wave guides, and strip lines.

9.6 Energy transport by electromagnetic waves

9.6.1 Power density

The energy the earth receives from the sun has traveled through space in the form of electromagnetic waves that satisfy Eq. (9.18). Where *is* this energy when it is traveling? How is it deposited in matter when it arrives?

In the case of a static electric field, such as the field between the plates of a charged capacitor, we found that the total energy of the system could be calculated by attributing to every volume element dv an amount of energy $(\epsilon_0 E^2/2)\,dv$ and adding it all up. Look back at Eq. (1.53). Likewise, the energy invested in the creation of a magnetic field could be calculated by assuming that every volume element dv in the field contains $(B^2/2\mu_0)\,dv$ units of energy. See Eq. (7.79). The idea that energy actually resides in the field becomes more compelling when we observe sunlight, which has traveled through a vacuum where there are no charges or currents, making something hot.

We can use this idea to calculate the rate at which an electromagnetic wave delivers energy. Consider a traveling plane wave (not a standing wave) of any form, at a particular instant of time. Assign to every infinitesimal volume element dv an amount of energy $(1/2)(\epsilon_0 E^2 + B^2/\mu_0)\,dv$, **E** and **B** being the electric and magnetic fields in that volume element at that instant. Since $1/\mu_0\epsilon_0 = c^2$, this energy can be written alternatively as $(\epsilon_0/2)(E^2 + c^2B^2)\,dv$. Now assume that this energy simply travels with speed c in the direction of propagation. In this way we can find the amount of energy that passes, per unit time, through unit area perpendicular to the direction of propagation.

Let us apply this to the sinusoidal wave described by Eqs. (9.22) and (9.23). At the instant $t = 0$, we have $E^2 = E_0^2 \sin^2 y$. Also, $B^2 = (E_0/c)^2 \sin^2 y$, since, as we subsequently found, B_0 must equal $\pm E_0/c$. The energy density in this field is therefore

$$\frac{\epsilon_0}{2}\left(E_0^2 \sin^2 y + c^2\left(\frac{E_0}{c}\right)^2 \sin^2 y\right) = \epsilon_0 E_0^2 \sin^2 y. \tag{9.33}$$

The mean value of $\sin^2 y$ averaged over a complete wavelength is just $1/2$. The mean energy density in the field is then $\epsilon_0 E_0^2/2$, and $\epsilon_0 E_0^2 c/2$ is the mean rate at which energy flows through a "window" of unit area perpendicular to the y direction. (This follows from the fact that, during a time t, a tube with length ct and cross-sectional area A is the volume that passes through a window with area A. The volume per area per time is therefore $(ct)A/At = c$.) We can say more generally that, for any continuous, repetitive wave, whether sinusoidal or not, the rate of energy flow per unit area, which we call the *power density* S, is given by

$$\boxed{S = \epsilon_0 \overline{E^2} c} \tag{9.34}$$

Here $\overline{E^2}$ is the mean square electric field strength, which was $E_0^2/2$ for the sinusoidal wave of amplitude E_0. S will be in joules per second per square meter, or equivalently watts per square meter, if E is in volts per meter and c is in meters per second.

In Gaussian units the formula for power density is

$$S = \frac{\overline{E^2} c}{4\pi}, \tag{9.35}$$

where S is in ergs per second per square centimeter if E is in statvolts per centimeter and c is in centimeters per second.

If you want to write Eq. (9.34) without reference to c, then substituting $c = 1/\sqrt{\mu_0 \epsilon_0}$ yields

$$\boxed{S = \frac{\overline{E^2}}{\sqrt{\mu_0/\epsilon_0}}} \tag{9.36}$$

This expression for S is based only on the physics that was known in 1861 when Maxwell wrote down his set of equations. That is, it invokes nothing about the nature of light; you can repeat the above derivation by using the expression for v in Eq. (9.26) without introducing the speed of light, c. The fact that $1/\sqrt{\mu_0 \epsilon_0}$ can indeed be replaced by c was conjectured by Maxwell in 1862, demonstrated experimentally by Hertz in 1888, and explained theoretically by Einstein in 1905 through his special theory of relativity. The last of these routes was the one we took in Chapters 5 and 6, where we showed that $\mu_0 = 1/\epsilon_0 c^2$.

The constant $\sqrt{\mu_0/\epsilon_0}$ in Eq. (9.36) has the dimensions of resistance, and its value is 376.73 ohms. Rounding it off to 377 ohms, we have a convenient and easily remembered formula:

$$\boxed{S(\text{watts/meter}^2) = \frac{\overline{E^2}(\text{volts/meter})^2}{377 \text{ ohms}}} \tag{9.37}$$

The units here reduce to: watts = volt^2/ohm, which are the same as in the standard $P = V^2/R$ expression for the power in an ordinary resistor. If you need help in remembering the number 377, it happens to be the number of radians per second in 60 hertz, and also the 14th Fibonacci number.

When the electromagnetic wave encounters an electrical conductor, the electric field causes currents to flow. This generally results in energy being dissipated within the conductor at the expense of the energy in the wave. The total reflection of the incident wave in Fig. 9.10 was a special case in which the conductivity of the reflecting surface was infinite.

If the resistivity of the reflector is not zero, the amplitude of the reflected wave will be less than that of the incident wave. Aluminum, for example, reflects visible light, at normal incidence, with about 92 percent efficiency. That is, 92 percent of the incident energy is reflected, the amplitude of the reflected wave being $\sqrt{0.92}$ or 0.96 times that of the incident wave. The lost 8 percent of the incident energy ends up as heat in the aluminum, where the current driven by the electric field of the wave encounters ohmic resistance. What counts, of course, is the resistivity of aluminum at the frequency of the light wave, in this case about $5 \cdot 10^{14}$ Hz. That may be somewhat different from the dc or low-frequency resistivity of the metal. Still, the reflectivity of most metals for visible light is essentially due to the same highly mobile conduction electrons that make metals good conductors of steady current. It is no accident that good conductors are generally shiny. But why clean copper looks reddish while aluminum looks "silvery" can't be explained without a detailed theory of each metal's electronic structure.

Energy can also be absorbed when an electromagnetic wave meets nonconducting matter. Little of the light that strikes a black rubber tire is reflected, although the rubber is an excellent insulator for low-frequency electric fields. Here the dissipation of the electromagnetic energy involves the action of the high-frequency electric field on the electrons in the molecules of the material. In the broadest sense, that applies to the absorption of light in everything around us, including the retina of the eye.

Some insulators transmit electromagnetic waves with very little absorption. The transparency of glass for visible light, with which we are so familiar, is really a remarkable property. In the purest glass fibers used for optical transmission of audio and video signals, a wave travels as much as a hundred kilometers, or more than 10^{11} wavelengths, before most of the energy is lost. However transparent a material medium may be, the propagation of an electromagnetic wave within the medium differs in essential ways from propagation through the vacuum. The matter interacts with the electromagnetic field. To take that interaction into account, Eq. (9.18) must be modified in a way that will be explained in Chapter 10.

9.6.2 The Poynting vector

With the help of Maxwell's equations, we can produce a more general version of the power density given in Eq. (9.34). That result was valid only for traveling waves. The present result will be valid for arbitrary electromagnetic fields. Furthermore, it will be valid as a function of time (and space), and not just as a time average. As above, our starting point will be the fact that the energy density of an electromagnetic field, which we label $\mathcal{U}$, is given by $\epsilon_0 E^2/2 + B^2/2\mu_0$. Consider the rate of change of $\mathcal{U}$. If we write E^2 and B^2 as $\mathbf{E} \cdot \mathbf{E}$ and $\mathbf{B} \cdot \mathbf{B}$, then

$$\frac{\partial \mathcal{U}}{\partial t} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{E} + \frac{1}{\mu_0} \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{B}. \tag{9.38}$$

The product rule works for vectors just as it does for regular functions, as you can check by explicitly writing out the Cartesian components. We can rewrite the time derivatives here with the help of the two "induction" Maxwell equations in free space, $\nabla \times \mathbf{B} = \mu_0\epsilon_0\,\partial\mathbf{E}/\partial t$ and $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$. This yields

$$\frac{\partial \mathcal{U}}{\partial t} = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \cdot \mathbf{E} - \frac{1}{\mu_0}(\nabla \times \mathbf{E}) \cdot \mathbf{B}. \tag{9.39}$$

The right-hand side of this expression conveniently has the same form as the right-hand side of the vector identity

$$\nabla \cdot (\mathbf{C} \times \mathbf{D}) = (\nabla \times \mathbf{C}) \cdot \mathbf{D} - (\nabla \times \mathbf{D}) \cdot \mathbf{C}. \tag{9.40}$$

Hence $\partial\mathcal{U}/\partial t = (1/\mu_0)\nabla \cdot (\mathbf{B} \times \mathbf{E})$. For reasons that will become clear, let's switch the order of $\mathbf{B}$ and $\mathbf{E}$, which brings in a minus sign. We then have

$$\frac{\partial \mathcal{U}}{\partial t} = -\frac{1}{\mu_0}\nabla \cdot (\mathbf{E} \times \mathbf{B}). \tag{9.41}$$

If we now define the *Poynting vector* $\mathbf{S}$ by

$$\boxed{\mathbf{S} \equiv \frac{\mathbf{E} \times \mathbf{B}}{\mu_0}} \qquad \text{(Poynting vector)}, \tag{9.42}$$

then we can write our result as

$$-\frac{\partial \mathcal{U}}{\partial t} = \nabla \cdot \mathbf{S}. \tag{9.43}$$

This equation should remind you of another one we have encountered. It has exactly the same form as the continuity equation,

$$-\frac{\partial \rho}{\partial t} = \nabla \cdot \mathbf{J}. \tag{9.44}$$

Therefore, just as $\mathbf{J}$ gives the current density (the flow of charge per time per area), we can likewise say that $\mathbf{S}$ gives the power density (the flow of energy per time per area). Equivalently, Eqs. (9.43) and (9.44) are the statements of conservation of energy and charge, respectively. Energy (or charge) can't just disappear; if the energy in a given region decreases, it must be the case that energy flowed out of that region, and into another region.

If you don't trust the analogy with $\mathbf{J}$, you can work with the integral form of Eq. (9.43). The integral of the energy density $\mathcal{U}$ over a given volume V is simply the total energy U contained in that volume. So we have

$$\frac{dU}{dt} = \frac{d}{dt}\int_V \mathcal{U}\,dv = \int_V \frac{\partial \mathcal{U}}{\partial t}\,dv = -\int_V \nabla \cdot \mathbf{S}\,dv = -\int_S \mathbf{S} \cdot d\mathbf{a}, \tag{9.45}$$

where we have used the divergence theorem. This shows that the rate of change of the energy in a given volume V equals the negative of the flux of the vector **S** outward through the closed surface S that bounds V. (Remember that $d\mathbf{a}$ is defined to be the outward-pointing normal.) The minus sign in Eq. (9.45) makes sense; a positive outward flux of **S** means that U is decreasing. Since Eq. (9.45) holds for an arbitrary closed volume, the natural interpretation of **S** is that it gives the rate of energy flow per area through any surface, closed or not.

The Poynting vector **S** gives the power density for an arbitrary electromagnetic field, not just for the special case of a traveling wave. For any electromagnetic field, at any given point at any instant in time, the direction of **S** gives the direction of the energy flow, and the magnitude of **S** gives the energy per time per area flowing through a small frame. The units of **S** are joules per second per square meter, or watts per square meter.

In the special case of a traveling wave (sinusoidal or not), we know from the third property listed in Section 9.4 that the velocity points in the direction of $\mathbf{E} \times \mathbf{B}$. This equals the direction of **S**, as must be the case. We also know that a traveling wave has **B** perpendicular to **E**, with $B = E/c$. The magnitude of **S** is therefore $S = E(E/c)/\mu_0$. Using $\mu_0 = 1/\epsilon_0 c^2$, we obtain $S = \epsilon_0 E^2 c$. This is the instantaneous power density. Its average value is simply $\overline{S} = \epsilon_0 \overline{E^2} c$, in agreement with Eq. (9.34). (In that equation we were using S, without the line over it, to denote the average power density.)

Interestingly, there can also be energy flow in a static electromagnetic field. Consider a very long stick with uniform linear charge density λ, moving with speed v in the longitudinal direction, say, rightward. Close to the stick and not too close to the ends, the stick creates **E** and **B** fields that are essentially static, with **E** pointing radially and **B** pointing tangentially. Their cross product is therefore nonzero, so the Poynting vector is nonzero. Hence there is energy flow, and it moves in the same direction as the stick moves (for either sign of λ), as you can show with the right-hand rule. The energy density at a given point (not too close to the ends) doesn't change, because energy flows into a given volume from the left at the same rate it flows out to the right. However, near the ends the fields are changing, so there *is* a net energy flow into or out of a given volume. (Think of a uniform caravan of cars moving along the highway. The density of cars changes only at points near the ends of the caravan.) The rightward flow of energy is consistent with the fact that the whole system is moving to the right.

The Poynting vector (named after John Henry Poynting) falls into a wonderful class of phonetically accurate theorems/results. Others are the Low energy theorem (after F. E. Low) dealing with low-energy photons, and the Schwarzschild radius of a black hole (after Karl Schwarzschild, whose last name means "black shield" in German).

Example (Energy flow into a capacitor) A capacitor has circular plates with radius R and is being charged by a constant current I. The electric field E between the plates is increasing, so the energy density is also increasing. This implies that there must be a flow of energy into the capacitor. Calculate the Poynting vector at radius r inside the capacitor (in terms of r and E), and verify that its flux equals the rate of change of the energy stored in the region bounded by radius r.

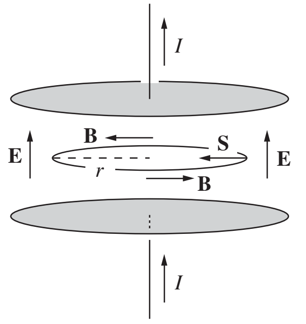

Figure 9.11.
The changing vertical electric field inside the capacitor induces a tangential magnetic field. The cross product of **E** and **B** yields an inward-pointing Poynting vector, consistent with the increasing energy density.

Solution If the Poynting vector is to be nonzero, there must be a nonzero magnetic field inside the capacitor. And indeed, because the electric field is changing, there is an induced magnetic field due to the $\nabla \times \mathbf{B} = \epsilon_0\mu_0\, \partial\mathbf{E}/\partial t$ Maxwell equation. If we integrate this equation over the area of a disk with radius r inside the capacitor (see Fig. 9.11) and use Stokes' theorem on the left-hand side, we obtain

$$\int \mathbf{B}\cdot d\mathbf{s} = \epsilon_0\mu_0 \frac{\partial E}{\partial t}\,(\text{area}) \implies B(2\pi r) = \epsilon_0\mu_0 \frac{\partial E}{\partial t}(\pi r^2)$$

$$\implies B = \frac{\epsilon_0\mu_0 r}{2}\frac{\partial E}{\partial t}. \tag{9.46}$$

This magnetic field points tangentially around the circle of radius r. Since **E** is increasing upward, **B** is directed counterclockwise when viewed from above, as you can check via the right-hand rule. The Poynting vector $\mathbf{S} = (\mathbf{E}\times\mathbf{B})/\mu_0$ then points radially inward everywhere on the circle of radius r. So the direction is correct; energy is flowing into the region bounded by radius r.

Let's now find the magnitude of **S**. Since **E** is perpendicular to **B**, the magnitude of **S** is

$$S = \frac{EB}{\mu_0} = \frac{E}{\mu_0}\left(\frac{\epsilon_0\mu_0 r}{2}\frac{\partial E}{\partial t}\right) = \frac{\epsilon_0 r}{2}E\frac{\partial E}{\partial t}. \tag{9.47}$$

To find the total energy per time (that is, the power) flowing past radius r, we must multiply S by the lateral area of the cylinder of radius r; that is, we must find the flux of S. If the separation between the plates is h, the lateral area is $2\pi rh$. The total power flowing into the cylinder of radius r is then

$$P = \left(\frac{\epsilon_0 r}{2}E\frac{\partial E}{\partial t}\right)2\pi rh = (\pi r^2 h)\epsilon_0 E\frac{\partial E}{\partial t} = \frac{d}{dt}\left((\text{volume})\frac{\epsilon_0 E^2}{2}\right) = \frac{dU}{dt}. \tag{9.48}$$

So the Poynting-vector flux does indeed equal the rate of change of the stored energy. In the special case where r equals the radius of the capacitor, R, we obtain the total power flowing into the capacitor. Note that S and P are largest at $r = R$, and zero at $r = 0$, as expected.

REMARK: You might be worried that although we found there to be a nonzero magnetic field inside the capacitor, we didn't take into account the resulting magnetic energy density, $B^2/2\mu_0$. We used only the electric $\epsilon_0 E^2/2$ part of the density. However, a constant current I implies a constant $d\sigma/dt$ (where $\pm\sigma$ are the charge densities on the plates), which in turn implies a constant $\partial E/\partial t$, which in turn implies a constant B, from Eq. (9.46). The magnetic energy density is therefore constant and thus doesn't affect the dU/dt in Eq. (9.48). We can therefore rightfully ignore it. On the other hand, if I *isn't* constant, then things are more complicated. However, for "everyday" rates of change of I, it is a very good approximation to say that the magnetic energy density in a capacitor is much smaller than the electric energy density; see Exercise 9.30.

At the end of Section 4.3 we mentioned that the energy flow in a circuit is due to the Poynting vector. We can now say more about this. There are two important parts to the energy flow. The first is the flow that yields the resistance heating. The current in a conducting wire is caused by a longitudinal $\mathbf{E}$ field inside the wire; recall $\mathbf{J} = \sigma\mathbf{E}$. Since the curl of $\mathbf{E}$ is zero, this same longitudinal $\mathbf{E}$ component must also exist right outside the surface of the wire. As you can show in Exercise 9.28, the Poynting-vector flux through a cylinder right outside the wire exactly accounts for the IV resistance heating.

The second part is the energy flow along the wire. As discussed at the end of Section 4.3, there are surface charges on the wire. These create an electric field perpendicular to the wire, which in turn creates a Poynting vector parallel to the wire, as you can verify. This gives an energy flow along the wire; see Galili and Goihbarg (2005). More generally, the energy flow need not be constrained to lie near the wire if the wire loops around in space. Energy can flow across open space too, from one part of a circuit to another; see Jackson (1996).

If there are other electric fields present in the system, there can be a third part to the energy flow, now *away* from the wire. See Problem 9.10.

9.7 How a wave looks in a different frame

A plane electromagnetic wave is traveling through the vacuum. Let $\mathbf{E}$ and $\mathbf{B}$ be the electric and magnetic fields measured at some place and time in F, by an observer in F. What field will be measured by an observer in a different frame who happens to be passing that point at that time? Suppose that frame F' is moving with speed v in the $\hat{\mathbf{x}}$ direction relative to F, with its axes parallel to those of F. We can turn to Eq. (6.74) for the transformations of the field components. Let us write them out again:

$$\begin{aligned} E'_x &= E_x, & E'_y &= \gamma(E_y - vB_z), & E'_z &= \gamma(E_z + vB_y); \\ B'_x &= B_x, & B'_y &= \gamma\big(B_y + (v/c^2)E_z\big), & B'_z &= \gamma\big(B_z - (v/c^2)E_y\big). \end{aligned} \tag{9.49}$$

of electrodynamics. It serves as a constraint on the sources (ρ and $\mathbf{J}$). They can't be just *any* old functions—they have to respect conservation of charge.[1]

The purpose of this chapter is to develop the corresponding equations for local conservation of energy and momentum. In the process (and perhaps more important) we will learn how to express the energy density and the momentum density (the analogs to ρ), as well as the energy "current" and the momentum "current" (analogous to $\mathbf{J}$).

8.1.2 ■ Poynting's Theorem

In Chapter 2, we found that the work necessary to assemble a static charge distribution (against the Coulomb repulsion of like charges) is (Eq. 2.45)

$$W_{\text{e}} = \frac{\epsilon_0}{2}\int E^2\,d\tau,$$

where $\mathbf{E}$ is the resulting electric field. Likewise, the work required to get currents going (against the back emf) is (Eq. 7.35)

$$W_{\text{m}} = \frac{1}{2\mu_0}\int B^2\,d\tau,$$

where $\mathbf{B}$ is the resulting magnetic field. This suggests that the total energy stored in electromagnetic fields, per unit volume, is

$$\boxed{u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right).} \tag{8.5}$$

In this section I will confirm Eq. 8.5, and develop the energy conservation law for electrodynamics.

Suppose we have some charge and current configuration which, at time t, produces fields $\mathbf{E}$ and $\mathbf{B}$. In the next instant, dt, the charges move around a bit. *Question*: How much work, dW, is done by the electromagnetic forces acting on these charges, in the interval dt? According to the Lorentz force law, the work done on a charge q is

$$\mathbf{F}\cdot d\mathbf{l} = q(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\mathbf{v}\,dt = q\mathbf{E}\cdot\mathbf{v}\,dt.$$

In terms of the charge and current densities, $q \to \rho d\tau$ and $\rho\mathbf{v} \to \mathbf{J}$,[2] so the rate at which work is done on all the charges in a volume $\mathcal{V}$ is

$$\frac{dW}{dt} = \int_{\mathcal{V}}(\mathbf{E}\cdot\mathbf{J})\,d\tau. \tag{8.6}$$

[1]The continuity equation is the *only* such constraint. Any functions $\rho(\mathbf{r}, t)$ and $\mathbf{J}(\mathbf{r}, t)$ consistent with Eq. 8.4 constitute possible charge and current densities, in the sense of admitting solutions to Maxwell's equations.

[2]This is a slippery equation: after all, if charges of both signs are present, the *net* charge density can be zero even when the current is *not*—in fact, this is the case for ordinary current-carrying wires. We should really treat the positive and negative charges separately, and combine the two to get Eq. 8.6, with $\mathbf{J} = \rho_+\mathbf{v}_+ + \rho_-\mathbf{v}_-$.

Evidently $\mathbf{E} \cdot \mathbf{J}$ is the work done per unit time, per unit volume—which is to say, the *power* delivered per unit volume. We can express this quantity in terms of the fields alone, using the Ampère-Maxwell law to eliminate $\mathbf{J}$:

$$\mathbf{E} \cdot \mathbf{J} = \frac{1}{\mu_0}\mathbf{E} \cdot (\nabla \times \mathbf{B}) - \epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t}.$$

From product rule 6,

$$\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B}).$$

Invoking Faraday's law ($\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$), it follows that

$$\mathbf{E} \cdot (\nabla \times \mathbf{B}) = -\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} - \nabla \cdot (\mathbf{E} \times \mathbf{B}).$$

Meanwhile,

$$\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(B^2), \quad \text{and} \quad \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(E^2), \tag{8.7}$$

so

$$\mathbf{E} \cdot \mathbf{J} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \frac{1}{\mu_0}\nabla \cdot (\mathbf{E} \times \mathbf{B}). \tag{8.8}$$

Putting this into Eq. 8.6, and applying the divergence theorem to the second term, we have

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau - \frac{1}{\mu_0}\oint_{\mathcal{S}} (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{a}, \tag{8.9}$$

where $\mathcal{S}$ is the surface bounding $\mathcal{V}$. This is **Poynting's theorem**; it is the "work-energy theorem" of electrodynamics. The first integral on the right is the total energy stored in the fields, $\int u\, d\tau$ (Eq. 8.5). The second term evidently represents the rate at which energy is transported out of $\mathcal{V}$, across its boundary surface, by the electromagnetic fields. Poynting's theorem says, then, that *the work done on the charges by the electromagnetic force is equal to the decrease in energy remaining in the fields, less the energy that flowed out through the surface*.

The *energy per unit time, per unit area*, transported by the fields is called the **Poynting vector**:

$$\boxed{\mathbf{S} \equiv \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}).} \tag{8.10}$$

Specifically, $\mathbf{S} \cdot d\mathbf{a}$ is the energy per unit time crossing the infinitesimal surface $d\mathbf{a}$—the energy *flux* (so $\mathbf{S}$ is the **energy flux density**).[3] We will see many

[3]If you're very fastidious, you'll notice a small gap in the logic here: We know from Eq. 8.9 that $\oint \mathbf{S} \cdot d\mathbf{a}$ is the total power passing through a *closed* surface, but this does not prove that $\int \mathbf{S} \cdot d\mathbf{a}$ is the power passing through any *open* surface (there could be an extra term that integrates to zero over all closed surfaces). This is, however, the obvious and natural interpretation; as always, the precise *location* of energy is not really determined in electrodynamics (see Sect. 2.4.4).

applications of the Poynting vector in Chapters 9 and 11, but for the moment I am mainly interested in using it to express Poynting's theorem more compactly:

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} u\,d\tau - \oint_{\mathcal{S}} \mathbf{S}\cdot d\mathbf{a}. \tag{8.11}$$

What if *no* work is done on the charges in $\mathcal{V}$—what if, for example, we are in a region of empty space, where there *is* no charge? In that case $dW/dt = 0$, so

$$\int \frac{\partial u}{\partial t}\,d\tau = -\oint \mathbf{S}\cdot d\mathbf{a} = -\int (\nabla\cdot\mathbf{S})\,d\tau,$$

and hence

$$\frac{\partial u}{\partial t} = -\nabla\cdot\mathbf{S}. \tag{8.12}$$

This is the "continuity equation" for *energy*—u (energy density) plays the role of ρ (charge density), and $\mathbf{S}$ takes the part of $\mathbf{J}$ (current density). It expresses local conservation of electromagnetic energy.

In *general*, though, electromagnetic energy by itself is *not* conserved (nor is the energy of the charges). Of course not! The fields do work on the charges, and the charges create fields—energy is tossed back and forth between them. In the overall energy economy, you must include the contributions of both the matter and the fields.

Example 8.1. When current flows down a wire, work is done, which shows up as Joule heating of the wire (Eq. 7.7). Though there are certainly *easier* ways to do it, the energy per unit time delivered to the wire can be calculated using the Poynting vector. Assuming it's uniform, the electric field parallel to the wire is

$$E = \frac{V}{L},$$

where V is the potential difference between the ends and L is the length of the wire (Fig. 8.1). The magnetic field is "circumferential"; at the surface (radius a) it has the value

$$B = \frac{\mu_0 I}{2\pi a}.$$

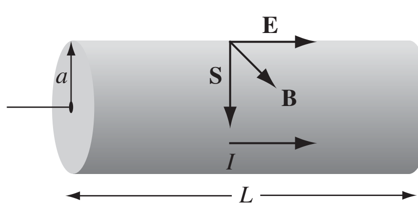

FIGURE 8.1

Accordingly, the magnitude of the Poynting vector is

$$S = \frac{1}{\mu_0}\frac{V}{L}\frac{\mu_0 I}{2\pi a} = \frac{VI}{2\pi a L},$$

and it points radially inward. The energy per unit time passing in through the surface of the wire is therefore

$$\int \mathbf{S}\cdot d\mathbf{a} = S(2\pi a L) = VI,$$

which is exactly what we concluded, on much more direct grounds, in Sect. 7.1.1.[4]

Problem 8.1 Calculate the power (energy per unit time) transported down the cables of Ex. 7.13 and Prob. 7.62, assuming the two conductors are held at potential difference V, and carry current I (down one and back up the other).

Problem 8.2 Consider the charging capacitor in Prob. 7.34.

(a) Find the electric and magnetic fields in the gap, as functions of the distance s from the axis and the time t. (Assume the charge is zero at $t = 0$.)

(b) Find the energy density u_{em} and the Poynting vector $\mathbf{S}$ in the gap. Note especially the *direction* of $\mathbf{S}$. Check that Eq. 8.12 is satisfied.

(c) Determine the total energy in the gap, as a function of time. Calculate the total power flowing into the gap, by integrating the Poynting vector over the appropriate surface. Check that the power input is equal to the rate of increase of energy in the gap (Eq. 8.9—in this case $W = 0$, because there is no charge in the gap). [If you're worried about the fringing fields, do it for a volume of radius $b < a$ well inside the gap.]

8.2 ■ MOMENTUM

8.2.1 ■ Newton's Third Law in Electrodynamics

Imagine a point charge q traveling in along the x axis at a constant speed v. Because it is moving, its electric field is *not* given by Coulomb's law; nevertheless, $\mathbf{E}$ still points radially outward from the instantaneous position of the charge (Fig. 8.2a), as we'll see in Chapter 10. Since, moreover, a moving point charge does not constitute a steady current, its magnetic field is *not* given by the Biot-Savart law. Nevertheless, it's a fact that $\mathbf{B}$ still circles around the axis in a manner suggested by the right-hand rule (Fig. 8.2b); again, the proof will come in Chapter 10.

[4]What about energy flow *down* the wire? For a discussion, see M. K. Harbola, *Am. J. Phys.* **78**, 1203 (2010). For a more sophisticated geometry, see B. S. Davis and L. Kaplan, *Am. J. Phys.* **79**, 1155 (2011).

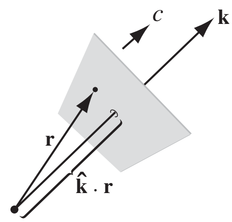

FIGURE 9.11

$$\mathbf{E}(\mathbf{r}, t) = E_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \delta)\,\hat{\mathbf{n}}, \tag{9.51}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{c} E_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \delta)(\hat{\mathbf{k}} \times \hat{\mathbf{n}}). \tag{9.52}$$

Problem 9.9 Write down the (real) electric and magnetic fields for a monochromatic plane wave of amplitude E_0, frequency ω, and phase angle zero that is (a) traveling in the negative x direction and polarized in the z direction; (b) traveling in the direction from the origin to the point (1, 1, 1), with polarization parallel to the xz plane. In each case, sketch the wave, and give the explicit Cartesian components of $\mathbf{k}$ and $\hat{\mathbf{n}}$.

9.2.3 ■ Energy and Momentum in Electromagnetic Waves

According to Eq. 8.5, the energy per unit volume in electromagnetic fields is

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right). \tag{9.53}$$

In the case of a monochromatic plane wave (Eq. 9.48)

$$B^2 = \frac{1}{c^2} E^2 = \mu_0 \epsilon_0 E^2, \tag{9.54}$$

so the *electric and magnetic contributions are equal*:

$$u = \epsilon_0 E^2 = \epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta). \tag{9.55}$$

As the wave travels, it carries this energy along with it. The energy flux density (energy per unit area, per unit time) transported by the fields is given by the Poynting vector (Eq. 8.10):

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}). \tag{9.56}$$

For monochromatic plane waves propagating in the z direction,

$$\mathbf{S} = c\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)\,\hat{\mathbf{z}} = cu\,\hat{\mathbf{z}}. \tag{9.57}$$

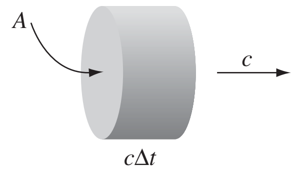

FIGURE 9.12

Notice that $\mathbf{S}$ is the energy density (u) times the velocity of the waves ($c\,\hat{\mathbf{z}}$)—as it *should* be. For in a time Δt, a length $c\,\Delta t$ passes through area A (Fig. 9.12), carrying with it an energy $uAc\,\Delta t$. The energy per unit time, per unit area, transported by the wave is therefore uc.

Electromagnetic fields not only carry energy, they also carry momentum. In fact, we found in Eq. 8.29 that the momentum density stored in the fields is

$$\mathbf{g} = \frac{1}{c^2}\mathbf{S}. \tag{9.58}$$

For monochromatic plane waves, then,

$$\mathbf{g} = \frac{1}{c}\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)\,\hat{\mathbf{z}} = \frac{1}{c}u\,\hat{\mathbf{z}}. \tag{9.59}$$

In the case of *light*, the wavelength is so short ($\sim 5 \times 10^{-7}$ m), and the period so brief ($\sim 10^{-15}$ s), that any macroscopic measurement will encompass many cycles. Typically, therefore, we're not interested in the fluctuating cosine-squared term in the energy and momentum densities; all we want is the *average* value. Now, the average of cosine-squared over a complete cycle[7] is $\frac{1}{2}$, so

$$\langle u\rangle = \frac{1}{2}\epsilon_0 E_0^2, \tag{9.60}$$

$$\langle \mathbf{S}\rangle = \frac{1}{2}c\epsilon_0 E_0^2\,\hat{\mathbf{z}}, \tag{9.61}$$

$$\langle \mathbf{g}\rangle = \frac{1}{2c}\epsilon_0 E_0^2\,\hat{\mathbf{z}}. \tag{9.62}$$

I use brackets, $\langle\ \rangle$, to denote the (time) average over a complete cycle (or *many* cycles, if you prefer). The average power per unit area transported by an electromagnetic wave is called the **intensity**:

$$I \equiv \langle S\rangle = \frac{1}{2}c\epsilon_0 E_0^2. \tag{9.63}$$

[7]There is a cute trick for doing this in your head: $\sin^2\theta + \cos^2\theta = 1$, and over a complete cycle the average of $\sin^2\theta$ is equal to the average of $\cos^2\theta$, so $\langle\sin^2\rangle = \langle\cos^2\rangle = 1/2$. More formally,

$$\frac{1}{T}\int_0^T \cos^2(kz - 2\pi t/T + \delta)\,dt = 1/2.$$

When light falls (at normal incidence) on a perfect absorber, it delivers its momentum to the surface. In a time Δt, the momentum transfer is (Fig. 9.12) $\Delta \mathbf{p} = \langle \mathbf{g} \rangle A c\, \Delta t$, so the **radiation pressure** (average force per unit area) is

$$P = \frac{1}{A}\frac{\Delta p}{\Delta t} = \frac{1}{2}\epsilon_0 E_0^2 = \frac{I}{c}. \tag{9.64}$$

(On a perfect *reflector* the pressure is *twice* as great, because the momentum switches direction, instead of simply being absorbed.) We can account for this pressure qualitatively, as follows: The electric field (Eq. 9.48) drives charges in the x direction, and the magnetic field then exerts on them a force $q(\mathbf{v} \times \mathbf{B})$ in the z direction. The net force on all the charges in the surface produces the pressure.[8]

Problem 9.10 The intensity of sunlight hitting the earth is about 1300 $\mathrm{W/m^2}$. If sunlight strikes a perfect absorber, what pressure does it exert? How about a perfect reflector? What fraction of atmospheric pressure does this amount to?

Problem 9.11 Consider a particle of charge q and mass m, free to move in the xy plane in response to an electromagnetic wave propagating in the z direction (Eq. 9.48—might as well set $\delta = 0$).

(a) Ignoring the magnetic force, find the velocity of the particle, as a function of time. (Assume the average velocity is zero.)

(b) Now calculate the resulting magnetic force on the particle.

(c) Show that the (time) average magnetic force is *zero*.

The problem with this naive model for the pressure of light is that the velocity is 90° out of phase with the fields. For energy to be absorbed, there's got to be some *resistance* to the motion of the charges. Suppose we include a force of the form $-\gamma m \mathbf{v}$, for some damping constant γ.

(d) Repeat part (a) (ignore the exponentially damped transient). Repeat part (b), and find the average magnetic force on the particle.[9]

Problem 9.12 In the complex notation there is a clever device for finding the time average of a product. Suppose $f(\mathbf{r}, t) = A \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta_a)$ and $g(\mathbf{r}, t) = B \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta_b)$. Show that $\langle fg \rangle = (1/2)\mathrm{Re}(\tilde{f}\tilde{g}^*)$, where the star denotes complex conjugation. [Note that this only works if the two waves have the same $\mathbf{k}$ and ω, but they need not have the same amplitude or phase.] For example,

$$\langle u \rangle = \frac{1}{4}\mathrm{Re}\left(\epsilon_0 \tilde{\mathbf{E}} \cdot \tilde{\mathbf{E}}^* + \frac{1}{\mu_0}\tilde{\mathbf{B}} \cdot \tilde{\mathbf{B}}^*\right) \quad \text{and} \quad \langle \mathbf{S} \rangle = \frac{1}{2\mu_0}\mathrm{Re}\left(\tilde{\mathbf{E}} \times \tilde{\mathbf{B}}^*\right).$$

Problem 9.13 Find all elements of the Maxwell stress tensor for a monochromatic plane wave traveling in the z direction and linearly polarized in the x direction (Eq. 9.48). Does your answer make sense? (Remember that $-\overleftrightarrow{\mathbf{T}}$ represents the momentum flux density.) How is the momentum flux density related to the energy density, in this case?

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[8] Actually, it's a little more subtle than this—see Prob. 9.11.

[9] C. E. Mungan, *Am. J. Phys.* **77**, 965 (2009). See also Prob. 9.34.